Unit 7: Applications of differentiation

7.1 Gradient

Teachers may use diagrams to introduce to students the geometrical meaning of the difference quotient $\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

Students should be convinced that as Δx approaches 0, the limiting value of $\frac{\Delta y}{\Delta x}$ will give the slope or gradient of the tangent to the curve at the point (x, $f(x)$). The concept of the normal to the curve $y = f(x)$ at (x, $f(x)$) can also be introduced. Students are expected to find the equations of tangents and normals to simple curves.

Students should learn to determine the region of increase and the region of decrease of a given function by analysing the changes in the sign of the derivative.

The convexity of a graph of a function can be introduced graphically first. Teachers should guide students through some simple functions (e.g. $y = e^x$, $y = \ln x$, $y = 3x^3 - 1$) to illustrate the concepts of the convexity of a graph.

Teachers may introduce to students that a point at which the convexity changes is called an inflection point. Students should be able to find an inflection point.

7.2 Maxima and minima

After acquiring the knowledge of gradients and convexity of a curve, turning points can now be discussed. With the help of diagrams, students should be reminded of the following points:

(i) the stationary points of a curve are not necessarily turning points

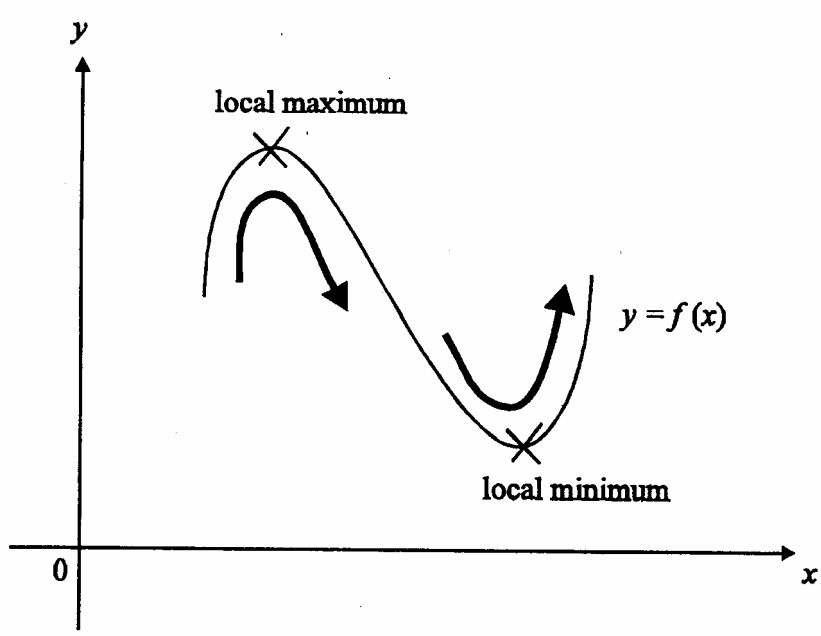
<u>Example 1</u>
For $f(x) = 2x^3$, $f'(x) = 6x^2$ so that $f'(0) = 0$ but $f'(0) > 0$ elsewhere, thus (0,0) is not a turning point.

(ii) the relative extrema are not necessarily the absolute extrema

Turning point occurring at points where the derivatives do not exist (e.g. $f(x) = 3x^{\frac{2}{3}}$) may also be introduced to abler students. It should also be noted that a maximum or a minimum value may occur even when $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$, e.g. $y = 3x^{\frac{2}{3}}$.

Two essential tests can be used to distinguish among maximum or minimum or inflection point,

(i) the first derivative test, which can easily be understood intuitively with the sketch of a portion of the graph; and



(ii) the second derivative test, teacher should emphasize that if $f''(a) = 0$, no conclusion can be drawn regarding the convexity at $(a,\ f(a))$. Another test such as the first derivative test is required to determine the nature of these particular critical points. Students should be taught to judge for themselves which tests to use.

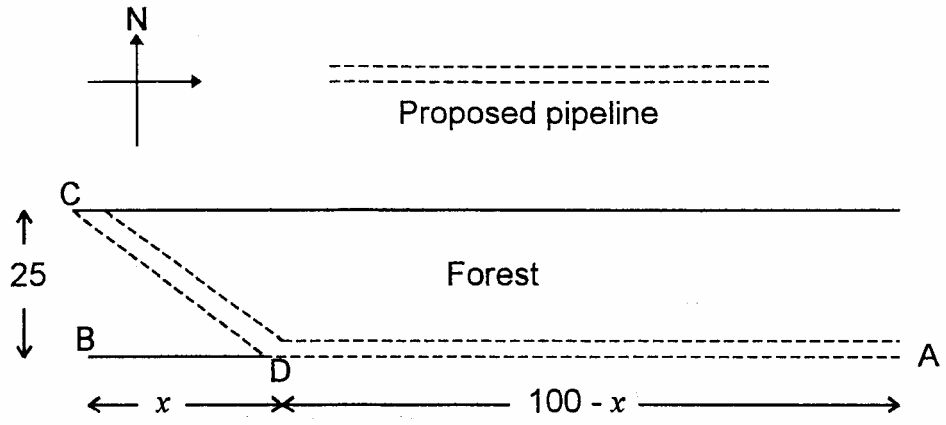
Example 2

$f(x) = 3x^4$ has a minimum point (0, 0), but $f''(0) = 12(0)^2 = 0$. So the second derivative test fails to work.

Very often in applied problems the domain is restricted. In searching for the absolute maximum or absolute minimum of a function defined on an interval, the behaviour of the function near the end points should also be considered. Real life examples can be used as illustrations.

Example 1

An oil company is planning to construct a pipeline to deliver crude oil from a well site *A* to a point *C* where the crude oil will be loaded on tankers. The proposed pipeline is shown in the figure below. Points *A* and *C* are on opposite sides of a dense forest which is 25 km wide and point *C* is 100 km west of *A*. The construction cost is \$100,000 per km along the edge of the forest and \$200,000 per km for the section crossing through the forest. Determine the crossing point *D* which will result in the minimum construction cost for the pipeline.



Example 2

A metal cylindrical vessel is designed to have a fixed capacity V cm^3. Find the height h cm of the cylinder in terms of the radius r cm if the cylinder is designed to have

(i) minimum surface area;

(ii) minimum welding edge.

Example 3

It is estimated that t years from now the population of a certain country will be $P(t) = \dfrac{150}{1+2e^{-0.02t}}$ million ($t \geq 0$). As time passes, the population of the country will approach a fixed number P_0 million. Find P_0. Find $P''(t)$ and hence determine the time, to the nearest year, at which the population will be growing most rapidly.

7.3 Simple curve-sketching

In determining the general shape of the graph of a function $y = f(x)$ the following attributes of the function:

(i) domain;

(ii) regions where $f(x)$ is increasing or decreasing;

(iii) local maxima and minima;

(iv) x-intercepts and y-intercepts;

(v) when $x \to \pm\infty$, the behaviour of the function;

(vi) convexity and inflection points of the curve.

Students should be able to sketch simple curves including graphs of polynomials and rational functions of' the form $\dfrac{ax+b}{cx+d}$. In addition, they should be able to find the horizontal and vertical asymptotes of the curves whenever they exist.

Example 1

A flu epidemic is spreading through a city. Based on a similar situation occurred in the past, the total number of people $N(t)$ who will be afflicted by the flu at time t is given by the equation $N(t) = -\frac{1}{3}t^3 + 10t^2 + 300t + 250$, where t is measured in days and $0 \le t \le 40$. Sketch the graph of $N(t)$ for $0 \le t \le 40$ and

(i) interpret the value of $N(0)$;
(ii) find the number of people expected to contract the flu at 10th day, 35th day;
(iii) determine the instantaneous rates at which the flu is expected to be spreading at $t = 11$ and $t = 12$;
(iv) find the maximum rate of contracting the flu; and
(v) interpret the inflection point of the graph.

Example 2

The function $f(x) = \frac{64(x+3)}{x+8}$ gives the number of correct words typed per minute that a trainee could accomplish after x practice sessions. Sketch the graph of $f(x)$ and predict the performance of a trainee if he/she has attended a large number of practice sessions.

7.4 Rate of change

The meaning that $\frac{dy}{dx}$ is the rate of change of y with respect to x should be taught. Examples involving various rates should be given.

Example 1

An oil tanker aground on a reef is leaking oil that forms a circular oil slick about 0.02m thick. It was found that the radius of the slick was increasing at 0.5m per minute when the radius was 50 m. Estimate the rates of increase of surface area and volume at which the oil is leaking from the tanker at that moment.

Example 2

An electronics company manufactures radios. Let the cost \$ C, revenue \$ R and profit \$ P of the company be given by the equations:

C = 5000 + 2x, $R = 10x - \frac{x^2}{1000}$ and $P = R - C$, where x is the number of radios produced per week. If the production is increasing at the rate of 500 radios per week when production is 2000 radios, find the rate of increase in the cost, revenue and profit.

Example 3

A 8 m ladder is placed against a vertical wall. If the bottom of the ladder is moving away from the wall at 2.5 m/s, at what rate is the top moving down when the top of the ladder is 6 m up the wall? Teachers may ask students to guess the answer before solving the problem.

Other examples such as finding the rate of change of the shadow when a man is moving away from a lamp-post, and the rate of water leaking from an inverted cone can also be introduced.

7.5 **Approximation**

Students should understand the geometrical meaning of differentials of the idea of $\Delta y \approx f'(x)\Delta x$ and $f(x+\Delta x) = f(x) + \Delta y \approx f(x) + f'(x)\Delta x$.

Students are expected to approximate values of a function close to special values and estimate simple error in the form $\frac{\mathrm{d}x}{x}$ by differentials.

<u>Example 1</u>
Find an approximate value of $\sqrt{9.03}$.

<u>Example 2</u>
In an experiment, the diameter, x cm, of a sphere is measured and the volume, V cm^3, calculated using the formula $V = \frac{1}{6}\pi x^3$. If the diameter is found to be 10 cm with a possible error of 0.02 cm, estimate the possible largest and smallest volumes of the sphere.

<u>Example 3</u>
During the process of manufacturing mechanical clocks, error is made in producing the length of the pendulum of the clock. The time, T seconds, taken for one complete swing of a pendulum, length ℓ m, is given by $T = 2\pi\sqrt{\frac{\ell}{g}}$, where g is a constant. If a 3% error is made in producing the length of a pendulum, estimate the percentage error in the value of T.